Part IIB - Economic Growth
Additional Notes

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Supervision 2: Substitutes or Complements?

When firms have a production function given by:

$$Y_i = L_{Y,i}^{1-\alpha} \sum_{j=1}^{A} x_{j,i}^{\alpha},$$

and a maximisation problem given as:

$$\max_{Y_i, L_{Y,i}, x_{j,i}} Y_i - wL_{Y,i} - \sum_{j=1}^{A} p_j x_{j,i},$$

we may solve as follows. Since there are constant returns to scale, firm ownership is unimportant (profits are zero in the final good sector), and we can assume a representative firm. This firm solves the problem:

$$\max_{L_Y, x_j} L_Y^{1-\alpha} \sum_{j=1}^{A} x_j^{\alpha} - w_Y L_Y - \sum_{j=1}^{A} p_j x_j,$$

with first order condition given by:

$$\frac{\partial \Pi}{\partial L_Y} = (1-\alpha) L_Y^{-\alpha} \sum_{j=1}^{A} x_j^{\alpha} - w_Y = 0,$$
$$\frac{\partial \Pi}{\partial x_j} = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} - p_j = 0.$$

These equations give the demand functions for both inputs as a function of prices. To show that x_j are not perfect substitutes we investigate the elasticity of substitution of two inputs, x_i and x_j, which may be defined as:

$$\sigma_{i,j} \equiv -\frac{\partial \ln \frac{x_i}{x_j}}{\partial \ln \frac{p_i}{p_j}}.$$

Demand for each input (using the first order condition above) may be written as:

$$x_i = L_Y \left[\frac{\alpha}{p_i}\right]^{\frac{1}{1-\alpha}},$$

and hence:

$$\begin{aligned}
\frac{x_i}{x_j} &= \frac{L_Y \left[\frac{\alpha}{p_i}\right]^{\frac{1}{1-\alpha}}}{L_Y \left[\frac{\alpha}{p_j}\right]^{\frac{1}{1-\alpha}}} = \left[\frac{p_j}{p_i}\right]^{\frac{1}{1-\alpha}}, \\
\ln \frac{x_i}{x_j} &= \frac{1}{1-\alpha} \ln \frac{p_j}{p_i} = -\frac{1}{1-\alpha} \ln \frac{p_i}{p_j}, \\
\frac{\partial \ln \frac{x_i}{x_j}}{\partial \ln \frac{p_i}{p_j}} &= -\frac{1}{1-\alpha},
\end{aligned}$$

and hence:

$$\sigma_{i,j} = \frac{1}{1-\alpha},$$

with $1 < \sigma_{i,j} < \infty$, as we have that $\alpha \in (0, 1)$. This shows that although the goods are substitutes (as $\sigma_{i,j} > 1$) they are not **perfect** substitutes for this producer (as this would imply $\sigma_{i,j} = \infty$).